International Journal of Theoretical Physics, Vol. 27, No. 1, 1988

Fourth-Order Orbital Equations in Stationary Weak Gravitational Fields

J. M. Gambi,[1] A. San Miguel,[2] and F. Vicente[2]

Received September 2, 1987

The equations of motion in fourth approximation for gravitational bodies are used to obtain orbital equations, first integrals, differential equations for the corresponding trajectories, and fourth-order contributions to the orbital motions in stationary weak gravitational fields.

1. INTRODUCTION

In Gambi *et al.*, (1987) the fourth-order equations of motion for material bodies in slow motion and weak fields were obtained using Synge's method of approximation (Synge, 1970). The contributions of the 4-force in these equations are given in terms of six gravitational potentials and combinations of them, which, generated by the material system, characterize the state of energy, stress, and rotation of the gravitational model under consideration. These contributions appear split into four + five components (four for the equations of motion and five for the equation of continuity), so that each one corresponds to an increasing and single, up to the fourth order of approximation.

The first two + three components have been studied extensively, using the equations of motion in third approximation (Hogan and McCrea, 1974; McCrea and O'Brien, 1978, O'Brien, 1979; Gambi, 1983, 1985; Gambi and San Miguel, 1986), and we have applied the fourth-order equations (Gambi *et al.*, 1987) to the study of the contributions of the remainder two + two to the orbital motions in static fields, so that only the two first characteristics, energy and stress, have been taken into account.

[1]CEME Department, University of Vermont, Burlington, Vermont.
[2]Departamento de Matemática Aplicada Fundamental, Facultad de Ciencias, Universidad de Valladolid, 47005-Valladolid, Spain.

0020-7748/88/0100-0107$06.00/0

Continuing this previous work, the aim of this paper is to derive the contributions corresponding to the potentials of rotation and, to this end, we shall consider the gravitational potentials generated by a massive body rotating steadily around an axis of rotation.

The plan of the paper is as follows: In Section 2 we obtain briefly, in parallel to the cited previous work, the equations of orbital motions in fourth approximation for stationary fields and then, to obtain first integrals of these equations in Section 3, suppose the generating body of the field has as axis of symmetry its axis of rotation. To obtain the equations for the trajectories we assume that the body has as a plane of symmetry the plane of its equator. Finally, using the canonical formulation, in Section 4 we obtain the fourth-order contributions corresponding to the mentioned potentials of rotation.

2. THE ORBITAL EQUATIONS

For details of Synge's method the reader is referred to Synge (1970). The general equations of motion in fourth approximation are

$$T^{ab}_{,b} + \kappa^{-1} \underset{3}{\hat{G}}{}^{ab}_{,b} = 0 \tag{1}$$

where T^{ab} is the energy tensor and $\underset{3}{\hat{G}}{}^{ab}$ is the truncated Einstein tensor corresponding to this approximation. In general $\underset{3}{\hat{G}}{}^{ab}$ is defined by

$$\underset{m}{\hat{G}}{}^{ab} = G^{ab} - L_{ab} \tag{2}$$

where L_{ab} is the linear part of the Einstein tensor G^{ab} for the metric $\underset{m}{\gamma}_{ab}$ corresponding to the mth approximation. If we assume that the field is weak, that is,

$$T^{\alpha\beta} = O(k^2), \qquad T^{\alpha 4} = O(k^{1/2}), \qquad T^{44} = O(k) \tag{3}$$

where k is the basis of the approximation, we have

$$\underset{m}{\hat{G}}{}^{ab} = \underset{m-1}{\hat{G}}{}^{ab} + O(k^{m+1}) \tag{4}$$

so that $\underset{m}{\hat{G}}{}^{ab}$ can be calculated using the metric corresponding to the $(n-1)$ approximation.

In our case, the metric in second approximation is given by (Synge, 1970)

$$\underset{2}{g}_{ab} = \delta_{ab} + \underset{2}{\gamma}_{ab} \tag{5}$$

where

$$\underset{2}{\gamma}_{\alpha\beta} = 2(V - K_{\sigma\sigma} + V^2)\delta_{\alpha\beta} + 4K_{\alpha\beta} + E_{\alpha\beta} + O(k^3) \tag{6}$$

$$\underset{2}{\gamma}_{\alpha 4} = 4i(W_\alpha + F_\alpha) + O(k^{7/2}) \tag{7}$$

$$\underset{2}{\gamma}_{44} = -2(V + K_{\sigma\sigma} - V^2) + O(k^3) \tag{8}$$

and

$$V = -RT^{44} = O(k) \tag{9}$$

$$W_\mu = -iRT^{\mu 4} = O(k^{3/2}) \tag{10}$$

$$K_{\mu\nu} = RT^{\mu\nu} = O(k^2) \tag{11}$$

$$E_{\mu\nu} = -(1/\pi)R(V_{,\mu}V_{,\nu} + 2VV_{,\mu\nu}) = O(k^2) \tag{12}$$

$$\begin{aligned} F_\mu = -(i/4\pi)R[&-3V_{,\mu}V_{,4} + 2i(V_{,\sigma}W_{\sigma,\mu} - V_{,\mu\sigma}W_\sigma) \\ &+ 2i(W_\mu \Box V - V \Box W_\mu) = O(k^{5/2}) \end{aligned} \tag{13}$$

where

$$Rf(x, t) = \int f(x', t - |x - x'|)|x - x'|^{-1}\, d_3x' \tag{14}$$

If the second term of (1) is written in the form

$$\underset{4}{X}_\mu = A_{\mu\beta\gamma}T^{\beta\gamma} + B_{\mu\beta 4}T^{\beta 4} + C_\mu T^{44} + O(k^5) \tag{15}$$

$$\underset{4}{X}_4 = A_{4\beta\gamma}T^{\beta\gamma} + B_{4\beta 4}T^{\beta 4} + C_4 T^{44} + O(k^5) \tag{16}$$

then, with the aid of the value for $\underset{3}{\hat{G}}^{ab}$ (see McCrea, 1981), by a straightforward calculation and taking into account (2) we have (Gambi *et al.*, 1987)

$$\begin{aligned} A_{\beta\mu\gamma} = (-4V - 4k_{\sigma\sigma} - V^2 - 2N)_{,\gamma}\delta_{\beta\mu} - (V_{,\mu} - K_{\sigma\sigma,\mu})\delta_{\beta\gamma} \\ -\tfrac{1}{2}(4K_{\beta\gamma} + E_{\beta\gamma})_{,\mu} - (4K_{\beta\mu} + E_{\beta\mu})_{,\gamma} + O(K^3) \end{aligned} \tag{17}$$

$$
\begin{aligned}
B_{\mu\beta 4} = & -4V_{,4}\delta_{\mu\beta} - 4K_{\sigma\sigma,4}\delta_{\mu\beta} - (V^2)_{,4}\delta_{\mu\beta} \\
& -2N_{,4}\delta_{\mu\beta} - 4i(W_{\beta,\mu} - W_{\mu,\beta}) - 4K_{\mu\beta,4} \\
& -8iV(W_{\beta,\mu} - W_{\mu,\beta}) - E_{\mu\beta,4} - 8iW_\mu V_{,\beta} \\
& +4i(F_{\beta,\mu} - F_{\mu,\beta}) + O(K^{7/2})
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
C_\mu = & -V_{,\mu} + 4VV_{,\mu} - 4iW_{\mu,4} - 4VK_{\sigma\sigma,\mu} \\
& +4K_{\mu\alpha}V_{,\alpha} - 8iVW_{\mu,4} - 12V^2V_{,\mu} \\
& +E_{\mu\alpha}V_{,\alpha} - 4iW_\mu V_{,4} - 4iF_{\mu,4} - 4(W_\sigma^2)_{,\mu} \\
& +\{R[-2\pi^{-1}W_{\sigma,\nu}W_{\nu,\sigma} - 2i\pi^{-1}V_{,\sigma}W_{\sigma,4} \\
& -4T^{44}K_{\sigma\sigma} - 4VT^{\sigma\sigma} - (2\pi)^{-1}(V_{,4})^2 \\
& +\pi^{-1}V_{,\sigma\nu}K_{\sigma\nu} + (4\pi)^{-1}V_{,\sigma\nu}E_{\sigma\nu} \\
& -\pi^{-1}VV_{,44} - \pi^{-1}VK_{\sigma\nu,\sigma\nu} + 4T^{44}V^2 \\
& -(4\pi)^{-1}VE_{\sigma\nu,\sigma\nu} + (2\pi)^{-1}VV_{,\sigma}^2]\}_{,\mu} + O(K^4)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
A_{4\beta\gamma} = & V_{,4}\delta_{\beta\gamma} - 4iW_{\beta,\gamma} + 4VV_{,4}\delta_{\beta\gamma} \\
& +8i(W_\beta V_{,\gamma} - VW_{\beta,\gamma}) - 4iV_{,\alpha}W_\alpha\delta_{\beta\gamma} \\
& -2K_{\beta\gamma,4} + \tfrac{1}{2}E_{\beta\gamma,4} - K_{\sigma\sigma,4} - 4iF_{\beta,\gamma} + O(K^3),
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
B_{4\beta 4} = & (-V^2 + 4K_{\sigma\sigma} - 2N + VE_{\sigma\sigma} + 8W_\sigma^2 - 4VK_{\sigma\sigma} + 8V^3)_{,\beta} \\
& +16W_\alpha(W_{\beta,\alpha} - W_{\alpha,\beta}) + 8iV_{,4}W_\beta \\
& -\pi^{-1}\{R[2(W_{\sigma,\nu})^2 + V_{,\sigma}(K_{\nu\nu,\sigma} - 2K_{\sigma\nu,\nu} \\
& -6iW_{\sigma,4} - \tfrac{3}{4}E_{\nu\nu,\sigma} + \tfrac{1}{2}E_{\sigma\nu,\nu}) \\
& +\Box V(4K_{\sigma\sigma} - 2V^2 - \tfrac{1}{2}E_{\sigma\sigma}) - \tfrac{1}{2}(V_{,4})^2 \\
& -6W_{\sigma,\nu}W_{\nu,\sigma} - 16\pi T^{44}(K_{\sigma\sigma} - V^2) \\
& -V_{,\sigma\nu}(2K_{\sigma\nu} + \tfrac{1}{2}E_{\sigma\nu}) \\
& -V(4V_{,44} + 4K_{\sigma\nu,\sigma\nu} + E_{\sigma\nu,\sigma\nu} \\
& +3(V_{,\sigma})^2 + 16\pi T^{\sigma\sigma})]\}_{,\beta} + O(K^{7/2})
\end{aligned} \tag{21}
$$

$$\begin{aligned}
C_4 = & -V_{,4}+3K_{\sigma\sigma,4}-2VV_{,4}-2N_{,4}+4iV_{\alpha}W_{\alpha} \\
& +2V^2V_{,4}-32W_{\mu}W_{\mu,4}+4iV_{,\alpha}F_{\alpha}+4iW_{\alpha}K_{\sigma\sigma,\alpha} \\
& +(VE_{\sigma\sigma})_{,4}+2(VK_{\sigma\sigma})_{,4}-8iVV_{,\alpha}W_{\alpha} \\
& -i[-12(W_{\sigma}^2)_{,t}+6(VK_{\sigma\sigma})_{,t}-6(V^3)_{,t} \\
& +\pi^{-1}\{R\{2(W_{\sigma,\nu})^2-4W_{\sigma,\nu}W_{\nu,\sigma} \\
& -12\pi T^{44}(K_{\sigma\sigma}-V^2)-V_{,\sigma}(-4iW_{\sigma,4} \\
& +K_{\nu\nu,\sigma}+2K_{\sigma\nu,\nu}-\tfrac{3}{4}E_{\nu\nu,\sigma}+\tfrac{1}{2}E_{\sigma\nu,\nu}) \\
& +V[12\pi T^{\sigma\sigma}-3V_{,44}-3K_{\sigma\nu,\sigma\nu}-\tfrac{3}{4}E_{\sigma\nu,\sigma\nu} \\
& -\tfrac{7}{2}(V_{,\sigma})^2]+V_{,\sigma\nu}K_{\sigma\nu}+\tfrac{1}{4}V_{,\sigma\nu}E_{\sigma\nu} \\
& -\Box V(4K_{\sigma\sigma}-2V^2-\tfrac{1}{2}E_{\sigma\sigma})\})_{,t}]+O(K^{9/2})
\end{aligned} \tag{22}$$

with the potential N in the expressions (17), (18), (21), and (22) given by

$$N=R(\rho V)=J(V\Box V)=O(K^2) \tag{23}$$

Now, adopting the Eulerian formalism for the first term of (1),

$$T^{\alpha\beta}=\rho u_{\alpha}u_{\beta}-S_{\alpha\beta}, \qquad T^{\alpha 4}=i\rho u_{\alpha}, \qquad T^{44}=-\rho<0 \tag{24}$$

where ρ, $S_{\alpha\beta}$, and u_{α} are the density, stress, and 3-velocity of the material system, respectively, from (17)-(22) we have

$$\rho\dot{u}_{\mu}+u_{\mu}(\dot{\rho}+\rho\theta)-S_{\mu\nu,\nu}=\rho V_{,\mu}+Y_{\mu}+Y'_{\mu}+O(K^5) \tag{25}$$

$$\dot{\rho}+\rho\theta=-\rho V_{,t}+Z_1+Z_2+O(K^{11/2}) \tag{26}$$

where

$$\begin{aligned}
Y_{\mu}= & (\rho u^2-S_{\sigma\sigma})V_{,\mu}-4(\rho u_{\mu}u_{\gamma}-S_{\mu\gamma})V_{,\gamma} \\
& +4\rho u_{\beta}(W_{\mu,\beta}-W_{\beta,\mu}-\delta_{\mu\beta}V_{,t}) \\
& +\rho D_{\mu}(-2V^2+K_{\sigma\sigma})+4\rho W_{\mu,t}
\end{aligned} \tag{27}$$

$$\begin{aligned}
Y'_{\mu}= & -(\rho u^2-S_{\sigma\sigma})K_{\nu\nu,\mu} \\
& +(\rho u_{\beta}u_{\gamma}-S_{\beta\gamma})[(2K_{\beta\mu}+\tfrac{1}{2}E_{\beta\mu})_{,\gamma} \\
& -2(2K_{\beta\mu}+\tfrac{1}{2}E_{\beta\mu})_{,\gamma}+\delta_{\mu\beta}(4K_{\sigma\sigma}-V^2-2N)_{,\gamma}] \\
& +\rho u_{\beta}[\{8[(W_{\beta,\mu}-W_{\mu,\beta})V+W_{\mu}V_{,\beta}]-4(F_{\beta,\mu}-F_{\mu,\beta}) \\
& -2(2K_{\mu\beta}+\tfrac{1}{2}E_{\mu\beta})_{,t}+\delta_{\mu\beta}(4K_{\sigma\sigma}-V^2-2N)_{,t}\} \\
& +\rho D_{\mu}[4(W_{\sigma}^2+V^3)+\mathscr{A}]-4\rho V(K_{\sigma\sigma,\mu}+2W_{\mu,t}) \\
& -\rho V_{,\sigma}(4K_{\mu\sigma}+E_{\mu\sigma})+4\rho F_{\mu,t}+4\rho W_{\mu}V_{,t}
\end{aligned} \tag{28}$$

$$\begin{aligned} Z_1 = & -(\rho u^2 - S_{\sigma\sigma}) V_{,t} - 4 W_{\beta,\gamma}(\rho u_\beta u_\gamma - S_{\beta\gamma}) \\ & + \rho u_\beta D_\beta (4K_{\sigma\sigma} - V^2 - 2N) \\ & + \rho D_t (3K_{\sigma\sigma} - N - V^2) - 4\rho V_{,\alpha} W_\alpha \end{aligned} \tag{29}$$

$$\begin{aligned} Z_2 = & -4(\rho u^2 - S_{\sigma\sigma})(V_{,\alpha} W_\alpha + V + V_{,t} + \tfrac{1}{4}K_{\sigma\sigma,t}) \\ & + 2(\rho u_\beta u_\gamma - S_{\beta\gamma})[4(W_\beta V_{,\gamma} - VW_{\beta\gamma,t} \\ & -\tfrac{1}{4}E_{\beta\gamma,t} - 2F_{\beta,\gamma}] \\ & + \rho u_\beta [16 W_\alpha (W_{\beta,\alpha} - W_{\alpha,\beta}) + 8 V_{,t} W_\beta)] \\ & + \rho u_\beta D_\beta (VE_{\sigma\sigma} + 8 W_\sigma^2 - 4 VK_{\sigma\sigma} + 8V^3 - \mathscr{B}) \\ & + \rho D_t \mathscr{C} + \rho V D_t (10 V^2 + E_{\sigma\sigma} - 4K_{\sigma\sigma}) \\ & + \rho V_{,t}(E_{\sigma\sigma} - 4K_{\sigma\sigma}) \\ & -2\rho(2V_{,\alpha}F_\alpha + 4W_\alpha W_{\alpha,t} + 2W_\sigma K_{\nu\nu,\sigma} - 4VV_{,\sigma}W_\sigma) \end{aligned} \tag{30}$$

with

$$\begin{aligned} \mathscr{A} = & R[-2\pi^{-1}(W_{\sigma,\nu} W_{\nu,\sigma}) - 2i\pi^{-1} V_{,\sigma} W_{\sigma,4} + 4\rho K_{\sigma\sigma} \\ & -4VT^{\sigma\sigma} - (2\pi)^{-1}(V_{,4})^2 + \pi^{-1} V_{,\sigma\nu} K_{\sigma\nu} \\ & + (4\pi)^{-1} V_{,\sigma\nu} E_{\sigma\nu} - \pi^{-1} VV_{,44} - \pi^{-1} VK_{\sigma\nu,\sigma\nu} \\ & -(4\pi)^{-1} VE_{\sigma\nu,\sigma\nu} + (2\pi)^{-1} VV_{,\sigma}^2 - 4\rho V^2] \end{aligned} \tag{31}$$

$$\begin{aligned} \mathscr{B} = & \pi^{-1} R[2(W_{\sigma,\nu})^2 + V_{,\sigma}(K_{\nu\nu,\sigma} + 2K_{\sigma\nu,\nu} - 6iW_{\sigma,4} \\ & -\tfrac{3}{4}E_{\nu\nu,\sigma} + \tfrac{1}{2}E_{\nu\sigma,\nu}) \\ & + \square V(4K_{\sigma\sigma} - 2V^2 - \tfrac{1}{2}E_{\sigma\sigma}) - 5 W_{\sigma,\nu} W_{\nu,\sigma} - \tfrac{1}{2}(V_{,4})^2 \\ & + 16\rho(K_{\sigma\sigma} - V^2) + V_{,\sigma\nu}(2K_{\sigma\nu} + \tfrac{1}{2}E_{\sigma\nu}) \\ & - V(4V_{,44} + 4K_{\sigma\nu,\nu\sigma} + E_{\sigma\nu,\nu\sigma} + 3(V_{,\sigma})^2 + 16\pi T^{\sigma\sigma}] \end{aligned} \tag{32}$$

$$\begin{aligned} C = & \pi^{-1} R\{2(W_{\sigma,\nu})^2 - 4W_{\sigma,\nu} W_{\nu,\sigma} + 12\rho\pi(K_{\sigma\sigma} - V^2) \\ & + V_{,\sigma} - 4iW_{\sigma,4} + K_{\nu\nu,\sigma} + 2K_{\sigma\nu,\nu} - \tfrac{3}{4}E_{\nu\nu,\sigma} + \tfrac{1}{2}E_{\nu\sigma,\nu} \\ & + V[-12\pi T^{\sigma\sigma} - 3V_{,44} - 3K_{\sigma\nu,\sigma\nu} - \tfrac{3}{4}E_{\sigma\nu,\sigma\nu} - \tfrac{7}{2}(V_{,\sigma})^2] \\ & + V_{,\sigma\nu} K_{\sigma\nu} + \tfrac{1}{4} V_{,\sigma\nu} E_{\sigma\nu} \\ & + \square V(4K_{\sigma\sigma} - 2V^2 - \tfrac{1}{2}E_{\sigma\sigma})\} \end{aligned} \tag{33}$$

Finally, if we consider in (25) and (26) the motion of two bodies with one of them very small with respect to the other and this last in steady motion, so that all the potentials are time-independent, we can ignore the self-potentials and stress in the small body, and then, from (25)-(33), we have

$$\begin{aligned}
\dot{u}_\mu = {} & V_{,\mu} - 4V_{,\gamma}u_\mu u_\gamma + V_{,\mu}u^2 - 4VV_{,\mu} \\
& + K_{\sigma\sigma,\mu} - 4W_{\beta,\mu}u_\beta + 4W_{\mu,\beta}u_\beta \\
& - K_{\sigma\sigma,\mu}u^2 + 2K_{\beta\gamma,\mu}u_\beta u_\gamma + \tfrac{1}{2}E_{\beta\gamma,\mu}u_\beta u_\gamma \\
& + 4V_{,\alpha}W_\alpha u_\mu + 4W_{\beta,\gamma}u_\beta u_\gamma u_\mu \\
& - 4K_{\beta\mu,\gamma}u_\beta u_\gamma - E_{\beta\mu,\gamma}u_\beta u_\gamma \\
& + 8VW_{\beta,\mu}u_\beta - 8VW_{\mu,\beta} - 4F_{\beta,\mu}u_\beta \\
& + 4F_{\mu,\beta}u_\beta + 8W_\mu V_{,\beta}u_\beta - 4VK_{\sigma\sigma,\mu} \\
& - 4K_{\mu\alpha}V_{,\alpha} + 12V^2V_{,\mu} - E_{\mu\alpha}V_{,\alpha} \\
& + 4(W^2)_{,\mu} + 2\pi^{-1}R(W_{\sigma,\nu}W_{\sigma,\nu})_{,\mu} \\
& + 4R(VT^{\sigma\sigma})_{,\mu} - 4R(\rho K_{\sigma\sigma})_{,\mu} \\
& - \pi^{-1}R(V_{,\sigma\nu}K_{\sigma\nu})_{,\mu} - (4\pi)^{-1}R(V_{,\sigma\nu}E_{\sigma\nu})_{,\mu} \\
& + \pi^{-1}R(VK_{\sigma\nu,\sigma\nu})_{,\mu} \\
& + (4\pi)^{-1}R(VE_{\sigma\nu,\sigma\nu})_{,\mu} - (2\pi)^{-1}R(VV_{,\sigma}^2)_{,\mu} \\
& + 4R(\rho V^2)_{,\mu} + O(K^4)
\end{aligned} \tag{34}$$

which are the orbital equations wanted. As can be seen, they are obtained as a particular case of the general equations obtained in Gambi *et al.* (1987) and contain the ones used there to derive the cited contributions in static weak fields [equations (110), (111), and (164), respectively, of Gambi *et al.* (1987)]. On the other hand, from direct inspection it can be seen that the 4-force on the small body is determined by the potentials V, W_μ, $K_{\mu\nu}$, $E_{\mu\nu}$, F_μ, and N [defined in (9)-(13) and (23), respectively] generated by the massive body. From (3) it is clear that they are $O(k)$, $O(k^{3/2})$, $O(k^2)$, $O(k^2)$, $O(k^{5/2})$, and $O(k^2)$, respectively; furthermore, as the field is stationary in this case, they are all instantaneous.

Among the diverse models of stationary weak fields there are two of interest. On one hand there are the gravitational fields generated by continuum bodies at rest, and on the other hand we have the fields generated by continuum bodies in steady motion, which obviously are more general because the first are not only stationary, but also static. From these last ones the most interesting case corresponds to that in which there is an axis of symmetry around which the body is rotating steadily. Now, as the potentials (9)-(13) and (23) which determine the motion of the material system by means of the general equations (25)-(26) also determine the gravitational field generated by the system, we can obtain the form of the field when this system is constituted by the two bodies described previously, in such a way that their motion is governed by equations (34), so that the field for the case in which the massive body has an axis of symmetry appears as a particular case. In fact, in accord with Synge's method (Synge, 1970), if the system moves according to equations (25)-(26), then we have for this approximation

$$\underset{3}{\gamma}^{*}_{\mu\nu} = -2J(\kappa T^{\mu\nu} + \underset{2}{\hat{G}}^{\mu\nu}) \tag{35}$$

$$\underset{3}{\gamma}^{*}_{\mu 4} = -2J(\kappa T^{\mu 4} + \underset{2}{\hat{G}}^{\mu 4}) \tag{36}$$

$$\underset{3}{\gamma}^{*}_{44} = -2J(\kappa T^{44} + \underset{2}{\hat{G}}^{44}) \tag{37}$$

where

$$\underset{3}{\gamma}^{*}_{ab} = \underset{3}{\gamma}_{ab} - \tfrac{1}{2}\delta_{ab}\underset{3}{\gamma}_{cc} \qquad \text{and} \qquad J = -(4\pi)^{-1}R$$

Now, eliminating in $\underset{3}{\hat{G}}^{ab}$ all the terms of $O(k^3)$, $O(k^{7/2})$, and $O(k^3)$ in $\underset{3}{\hat{G}}^{\mu\nu}$, $\underset{3}{\hat{G}}^{\mu 4}$, and $\underset{3}{\hat{G}}^{44}$, respectively, we have after a straightforward and tedious calculation

$$\begin{aligned}
\underset{3}{\gamma}_{\mu\nu} = {} & 2(V - K_{\sigma\sigma})\delta_{\nu\nu} + 4K_{\mu\nu} \\
& + \pi^{-1}R[-(V_{,\mu}V_{,\nu}) - \delta_{\mu\nu}(V_{,\sigma})^2 - 2(VV_{,\mu\nu}) - \delta_{\mu\nu}(V\Box V) \\
& - (V_{,\mu}K_{\sigma\sigma,\nu} + V_{,\nu}K_{\sigma\sigma,\mu}) + 2\delta_{\mu\nu}(V_{,\alpha}K_{\sigma\sigma,\alpha}) \\
& + V_{,\sigma}(-4K_{\mu\nu,\sigma} + 2K_{\mu\nu,\sigma} + 2K_{\mu\sigma,\nu}) + 4(W_{\sigma,\mu}W_{\sigma,\nu}) \\
& + 4(W_{\mu,\sigma}W_{\nu,\sigma}) + 2iV_{,4}(W_{\mu,\nu} + W_{\nu,\mu}) - i(V_{,4}W_{\sigma,\sigma}) \\
& - 2\delta_{\mu\nu}(VV_{,44}) + 8(W_\sigma W_{\sigma,\mu\nu}) - \tfrac{3}{2}\delta_{\mu\nu}(VK_{\sigma\sigma,\alpha\alpha}) \\
& + \tfrac{1}{2}\delta_{\mu\nu}(V_{,\alpha\alpha}K_{\sigma\sigma}) - 2\delta_{\mu\nu}(V_{,\sigma\alpha}K_{\sigma\alpha}) + 4(V\Box K_{\mu\nu}) \\
& + 2i(V_{,\mu 4}W_\nu + V_{,\nu 4}W_\mu) - 2(K_{\mu\nu}\Box V) + \tfrac{3}{2}\delta_{\mu\nu}(V\Box K_{\sigma\sigma}) \\
& + 2V(K_{\sigma\mu,\nu\sigma} + K_{\sigma\nu,\mu\sigma} - K_{\mu\nu,\sigma\sigma} - K_{\sigma\sigma,\mu\nu}) \\
& + 2(V_{,\mu\sigma}K_{\nu\sigma} + V_{,\nu\sigma}K_{\mu\sigma} - V_{,\mu\nu}K_{\sigma\sigma}) - 2\delta_{\mu\nu}V(V_{,\sigma})^2 \\
& - 2iV(W_{\mu,\nu 4} + W_{\nu,\mu 4}) - 4W_\sigma(W_{\mu,\nu\sigma} + W_{\nu,\sigma\mu}) \\
& + 4(W_\mu \Box W_\nu + W_\nu \Box W_\mu) + \tfrac{5}{2}\delta_{\mu\nu}(K_{\sigma\sigma}\Box V) + 2(VV_{,\mu}V_{,\nu})
\end{aligned}$$

$$-4(V^2V_{,\mu\nu})-(V_{,\sigma}E_{\mu\nu,\sigma})-\tfrac{1}{4}(V_{,\mu}E_{\sigma\sigma,\nu}+V_{,\nu}E_{\sigma\sigma,\mu})$$
$$+\tfrac{1}{2}V_{,\sigma}(E_{\mu\sigma,\nu}+E_{\nu\sigma,\mu})-\tfrac{1}{2}\delta_{\mu\nu}(V_{,\sigma\alpha}E_{\sigma\alpha})-2(V_{,\mu\nu}E_{\sigma\sigma})$$
$$+\tfrac{1}{2}(V_{,\mu\sigma}E_{\nu\sigma})+\tfrac{1}{2}(V_{,\nu\sigma}E_{\mu\sigma})-\tfrac{1}{2}(E_{\mu\nu}\Box V)$$
$$-\tfrac{1}{4}\delta_{\mu\nu}(E_{\sigma\sigma}\Box V)+\tfrac{1}{2}V(E_{\mu\sigma,\nu\sigma}+E_{\nu\sigma,\sigma\mu}-E_{\sigma\sigma,\mu\nu})$$
$$-4\delta_{\mu\nu}(W_\sigma W_{\sigma,\alpha\alpha})-2\delta_{\mu\nu}V(K_{\sigma\alpha,\sigma\alpha})+4\delta_{\mu\nu}(W_\sigma W_{\alpha,\sigma\alpha})$$
$$+2i\delta_{\mu\nu}V(W_{\alpha,\alpha4})+2\delta_{\mu\nu}(V^2V_{,\alpha\alpha})+\tfrac{1}{4}\delta_{\mu\nu}(V_{,\alpha\alpha}E_{\sigma\sigma})$$
$$-\tfrac{1}{4}V(2E_{\alpha\sigma,\alpha\sigma}-E_{\sigma\sigma,\alpha\alpha})\delta_{\mu\nu}-\delta_{\mu\nu}(V^2\Box V)]+O(K^4) \tag{38}$$

$$\underset{3}{\gamma}_{\mu4}=4iW_\mu-3\pi^{-1}(V_{,\mu}V_{,4})$$
$$+\pi^{-1}R[2i(V_{,\eta}W_{\eta,\mu}-W_\eta V_{\eta\mu})+2i(W_\mu\Box V-V\Box W_\mu)$$
$$+4i(W_\mu V_{,44}-VW_{\mu,44})+(K_{\eta\eta,\mu}V_{,4}+K_{\eta\eta,4}V_{,\mu})$$
$$+2i(W_\mu\Box K_{\eta\eta}+3K_{\eta\eta}+3K_{\eta\eta}\Box W_\mu)-4i\Box(K_{\mu\eta}W_\eta)$$
$$+4iK_{\eta\sigma}W_{\eta,\mu\sigma}-2i(K_{\eta\eta}W_\sigma)_{,\mu\sigma}-2(VK_{\eta\mu})_{,\eta4}$$
$$+4i(K_{\mu\eta}W_\sigma-W_\mu K_{\eta\sigma})_{,\eta\sigma}+2(W_\eta W_\eta)_{,\mu4}+4(W_\mu W_{\eta,4})_{,\eta}$$
$$-\tfrac{3}{4}(V_{,\mu}H_{,4}+V_{,4}+V_{,4}H_{,\mu})-\tfrac{1}{2}(W_\mu\Box H-H\Box W_\mu)$$
$$-\tfrac{1}{2}i(W_{\eta,\mu}H_{,\eta}-H_{,\mu\eta}W_\eta)+\tfrac{1}{2}(\Omega_\mu\Box V-V\Box\Omega_\mu)$$
$$+\tfrac{1}{2}i(\Omega_{\eta,\mu}V_{,\eta}-V_{,\mu\eta}\Omega_\eta)+4iVV_{,\eta}(W_{\mu,\eta}-W_{\eta,\mu})$$
$$+8iV(V\Box W_\mu-W_\mu\Box V)+2iV_{,\eta}(2V_{,\mu}W_\eta-3V_{,\eta}W_\mu)$$
$$+8(VV_{,\mu}V_{,4}+iVV_{,\mu\eta}W_\eta)]+O(K^{9/2}) \tag{39}$$

$$\underset{3}{\gamma}_{44}=2V-2K_{\sigma\sigma}+2V^2-8W_\sigma^2-4V^3+4VK_{\sigma\sigma}$$
$$+\pi^{-1}R[2(V_{,\sigma\nu}K_{\sigma\nu})-4(W_{\sigma,\nu}W_{\nu,\sigma})-4i(V_{,\sigma}W_{\sigma,4})$$
$$+8\pi(\rho K_{\sigma\sigma})-8\pi(VT^{\sigma\sigma})-(V_{,4})^2+\tfrac{1}{2}(V_{,\sigma\nu}E_{\sigma\nu})$$
$$-2(VV_{,44})-2(VK_{\sigma\nu,\sigma\nu})-\tfrac{1}{2}(VE_{\sigma\nu,\sigma\nu})+(VV_{,\sigma}^2)$$
$$-8\pi(\rho V^2)]+O(K^4) \tag{40}$$

where

$$H=4V^2+(2\pi)^{-1}R(V_{,\sigma}V_{,\sigma}) \tag{41}$$

$$\Omega_\mu=3i\pi^{-1}R(V_{,\mu}V_{,4})$$
$$+2\pi^{-1}R(V_{,\sigma}W_{\sigma,\mu}-V_{,\mu\sigma}W_\sigma-V\Box W_\mu+W_\mu\Box V) \tag{42}$$

and therefore the field is given by

$$g_{ab} = \delta_{ab} + \underset{3}{\gamma_{ab}} \tag{43}$$

so that the field equations are satisfied with an error $O(k^5)$. Obviously the metric (43) is simpler if the field is stationary, because all the terms in (38)-(40) are independent of x_4 in this case. If, in particular, the field is generated by a body with an axis of symmetry about which it is rotating, we have even another simplification, because assuming the axis of symmetry is the axis $0x_3$ (which obviously is not an essential restriction), the following conditions are satisfied in this case:

$$\rho = \rho(r, x_3), \qquad S_{\mu\nu} = S_{\mu\nu}(r, x_3) = O(K^2)$$

$$u_1 = -\frac{u}{r}x_2, \qquad u_2 = \frac{u}{r}x_1, \qquad u_3 = 0, \qquad u_4 = i, \qquad u = u(r, x_3) \tag{44}$$

where $u^2 = u_1^2 + u_2^2$ and $r^2 = r_1^2 + r_2^2$. In the next section we shall obtain first integrals of equations (34), but, since for the integral of energy we only need the field to be stationary, we shall use (43) taking into account only the fact that all the components are independent of x_4. The integral of angular momentum requires that the field have an axis of symmetry, and so, to obtain it, we shall use the conditions (44).

3. FIRST INTEGRALS AND TRAJECTORIES

To obtain first integrals of equations (34) let us first consider the field components (38)-(40) when the field is stationary. In general the Lagrangian L of a test particle is given by

$$L = (-g_{\mu\nu}\dot{x}_\mu\dot{x}_\nu - 2ig_{\mu 4}\dot{x}_\mu - g_{44})^{1/2} \tag{45}$$

so that for its motion we have the equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = 0 \tag{46}$$

or, equivalently,

$$\ddot{x}_\mu + \Gamma^\mu_{mn}\dot{x}_n = \lambda x_\mu, \qquad \Gamma^4_{mn}\dot{x}_m\dot{x}_n = i\lambda \tag{47}$$

where λ is a Lagrange multiplier, $\dot{x}_\mu = u_\mu = O(K^{1/2})$ as before, and Γ^h_{mn} are the Christoffel symbols of the second kind for the metric g_{ab}.

As the first three equations are equivalent to

$$\dot{u}_\mu + \Gamma^\mu_{\alpha\beta}u_\alpha u_\beta + 2i\Gamma^\mu_{\alpha 4}u_\alpha - \Gamma^\mu_{44} = \lambda u_\mu \tag{48}$$

then, if the field is stationary, we have

$$\begin{aligned}
\Gamma^{\mu}_{\alpha\beta} &= \tfrac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta}g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) + \tfrac{1}{2}g^{\mu 4}(g_{\alpha 4,\beta} + g_{\beta 4,\alpha}) \\
\Gamma^{\mu}_{\alpha 4} &= \tfrac{1}{2}g^{\mu\sigma}(g_{4\sigma,\alpha} - g_{\alpha 4,\sigma}) + \tfrac{1}{2}{}^{\mu 4}g_{44,\alpha} \\
\Gamma^{\mu}_{44} &= -\tfrac{1}{2}g^{\mu\sigma}g_{44,\sigma}
\end{aligned} \tag{49}$$

so that, carrying (49) to (48), we have

$$\begin{aligned}
&\dot{u}_{\mu} + \tfrac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) \\
&\quad + \tfrac{1}{2}g^{\mu 4}(g_{\alpha 4,\beta} + g_{\beta 4,\alpha})u_{\alpha}u_{\beta} \\
&\quad + 2i \cdot \tfrac{1}{2}g^{\mu\sigma}(g_{4\sigma,\alpha} - g_{4\alpha,\sigma}) \\
&\quad + \tfrac{1}{2}g^{\mu 4}g_{44,\alpha}u_{\alpha} + \tfrac{1}{2}g^{\mu\sigma}g_{44,\sigma} = \lambda u_{\mu}
\end{aligned} \tag{50}$$

Now, if we suppose that the metric is (43), then, from (49), we have

$$\begin{aligned}
&\dot{u}_{\mu} + \tfrac{1}{2}\underset{3}{\gamma_{44,\mu}} - \tfrac{1}{2}\underset{3}{\gamma_{\varepsilon\mu}}\,\underset{3}{\gamma_{44,\varepsilon}} + \tfrac{1}{2}\underset{3}{\gamma_{\mu\sigma}}\,\underset{3}{\gamma_{\sigma\eta}}\,\underset{3}{\gamma_{44,\eta}} \\
&\quad + \tfrac{1}{2}(\underset{3}{\gamma_{\alpha\mu,\beta}} + \underset{3}{\gamma_{\beta\mu,\alpha}} - \underset{3}{\gamma_{\alpha\beta,\mu}} + \underset{3}{\gamma_{\mu\varepsilon}}\,\underset{3}{\gamma_{\alpha\beta,\varepsilon}} \\
&\quad - \underset{3}{\gamma_{\sigma\mu}}\,\underset{3}{\gamma_{\alpha\sigma,\beta}} - \underset{3}{\gamma_{\sigma\mu}}\,\underset{3}{\gamma_{\beta\sigma,\alpha}})u_{\alpha}u_{\beta} \\
&\quad + i(\underset{3}{\gamma_{4\mu,\alpha}} - \underset{3}{\gamma_{\alpha 4,\mu}} - \underset{3}{\gamma_{\mu\sigma}}\,\underset{3}{\gamma_{4\sigma,\mu}} \\
&\quad - \underset{3}{\gamma_{\mu 4}}\,\underset{3}{\gamma_{44,\alpha}} + \underset{3}{\gamma_{\mu\varepsilon}}\,\underset{3}{\gamma_{\alpha 4,\varepsilon}})u_{\alpha} = \lambda u_{\mu}
\end{aligned} \tag{51}$$

so that, going back to the fourth equation (47), we have

$$\begin{aligned}
\Gamma^{4}_{\alpha\beta} &= \tfrac{1}{2}g^{4\varepsilon}(g_{\alpha\varepsilon,\beta} + g_{\beta\varepsilon,\alpha} - g_{\alpha\beta,\varepsilon}) \\
&\qquad + \tfrac{1}{2}g^{44}(g_{4\alpha,\beta} + g_{4\beta,\alpha}) \\
\Gamma^{4}_{\alpha 4} &= \tfrac{1}{2}g^{4\varepsilon}(g_{4\varepsilon,\alpha} - g_{4\alpha,\varepsilon}) + \tfrac{1}{2}g^{44}g_{44,\alpha} \\
\Gamma^{4}_{44} &= -\tfrac{1}{2}g^{4\alpha}g_{44,\alpha}
\end{aligned} \tag{52}$$

and so

$$\begin{aligned}
&\tfrac{1}{2}g^{4\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) \\
&\quad + \tfrac{1}{2}g^{44}(g_{4\alpha,\beta} + g_{4\beta,\alpha})u_{\alpha}u_{\beta} \\
&\quad + 2i \cdot \tfrac{1}{2}g^{4\sigma}(g_{4\sigma,\alpha} - g_{4\alpha,\sigma}) + \tfrac{1}{2}g^{44}g_{44,\alpha}u_{\alpha} \\
&\quad + \tfrac{1}{2}g^{4\sigma}g_{44,\sigma} = i\lambda
\end{aligned} \tag{53}$$

so that

$$
\begin{aligned}
&\tfrac{1}{2}(-\underset{3}{\gamma}_{4\sigma})(\underset{3}{\gamma}_{\alpha\sigma,\beta}+\underset{3}{\gamma}_{\beta\sigma,\alpha}-\underset{3}{\gamma}_{\alpha\beta,\sigma})\\
&\quad+\tfrac{1}{2}(1-\underset{3}{\gamma}_{44})(\underset{3}{\gamma}_{4\alpha,\beta}+\underset{3}{\gamma}_{4\beta,\alpha})u_\alpha u_\beta\\
&\quad+2i\cdot\tfrac{1}{2}(-\underset{3}{\gamma}_{4\sigma})(\underset{3}{\gamma}_{4\sigma,\alpha}-\underset{3}{\gamma}_{4\alpha,\sigma})+\tfrac{1}{2}(1-\underset{3}{\gamma}_{44})\underset{3}{\gamma}_{44,\alpha}u_\alpha\\
&\quad+\tfrac{1}{2}(-\underset{3}{\gamma}_{4\sigma})(\underset{3}{\gamma}_{44,\sigma})=i\lambda
\end{aligned}
\tag{54}
$$

with which, from (51) and (54), we have

$$
\begin{aligned}
&\dot{u}_\mu+\tfrac{1}{2}(\underset{3}{\gamma}_{\alpha\mu,\beta}+\underset{3}{\gamma}_{\beta\mu,\alpha}+\underset{3}{\gamma}_{\alpha\beta,\mu}-\underset{3}{\gamma}_{\mu\sigma}\underset{3}{\gamma}_{\alpha\sigma,\beta}-\underset{3}{\gamma}_{\mu\sigma}\underset{3}{\gamma}_{\beta\sigma,\alpha}+\underset{3}{\gamma}_{\mu\sigma}\underset{3}{\gamma}_{\alpha\beta,\sigma})\\
&\quad-\tfrac{1}{2}(\underset{3}{\gamma}_{4\mu}\underset{3}{\gamma}_{4\alpha,\beta}+\underset{3}{\gamma}_{4\mu}\underset{3}{\gamma}_{4\beta,\alpha})u_\alpha u_\beta\\
&\quad+i(\underset{3}{\gamma}_{4\mu,\alpha}-\underset{3}{\gamma}_{4\alpha,\mu}-\underset{3}{\gamma}_{\mu\sigma}\underset{3}{\gamma}_{4\sigma,\alpha}+\underset{3}{\gamma}_{\mu\sigma}\underset{3}{\gamma}_{4\alpha,\sigma}-\underset{3}{\gamma}_{\mu 4}\underset{3}{\gamma}_{44,\alpha})u_\alpha\\
&\quad+\tfrac{1}{2}(\underset{3}{\gamma}_{44,\mu}-\underset{3}{\gamma}_{\mu\sigma}\underset{3}{\gamma}^{4,\sigma})\\
&=\underset{3}{\gamma}_{44,\beta}u_\beta u_\mu-\underset{3}{\gamma}_{44}\underset{3}{\gamma}_{44,\alpha}u_\alpha u_\mu-i\underset{3}{\gamma}_{\alpha 4,\beta}u_\alpha u_\beta u_\mu\\
&\quad+\tfrac{1}{2}i\underset{3}{\gamma}_{4\sigma}\underset{3}{\gamma}_{44,\sigma}u_\mu+O(K^4)
\end{aligned}
\tag{55}
$$

From these equations it can be seen that if (43) is substituted in (55), having eliminated in (38)-(40) all the terms that are taken as derivatives with respect to x_4, then we obtain equations (34). Thus, it is clear that if the field (43) is stationary, we have

$$L-\dot{x}_\mu\,\partial L/\partial\dot{x}=1+E \tag{56}$$

or, which is the same,

$$L^{-1}(-g_{\mu 4}\dot{x}_\mu+g_{44})-1=E \tag{57}$$

where L is the Lagrangian associated with the metric (43),

$$L=1-(u^2+\underset{3}{\gamma}_{\mu\nu}u_\mu u_\nu+2i\underset{3}{\gamma}_{\mu 4}u_\mu-\underset{3}{\gamma}_{44})^{1/2} \tag{58}$$

Now, since from (58) we have

$$
\begin{aligned}
L^{-1} = 1 &+ \tfrac{1}{2}(u^2 + \underset{3}{\gamma}_{\mu\nu} u_\mu u_\nu + 2i\underset{3}{\gamma}_{\mu 4} u_\mu - \underset{3}{\gamma}_{44}) \\
&+ \tfrac{3}{8}(u^4 + \underset{3}{\gamma}_{44}^2 + 2u^2 \underset{3}{\gamma}_{\mu\nu} u_\mu u_\nu + 4iu^2 \underset{3}{\gamma}_{\mu 4} u_\mu - 2u^2 \underset{3}{\gamma}_{44} \\
&- 2\underset{3}{\gamma}_{44} \underset{3}{\gamma}_{\mu\nu} u_\mu u_\nu - 4i\underset{3}{\gamma}_{\mu 4} \underset{3}{\gamma}_{44u_\mu}) \\
&+ \tfrac{5}{16}(u^6 - 3u^4 \underset{3}{\gamma}_{44} + 3u^2 \underset{3}{\gamma}_{44}^2 - \underset{3}{\gamma}_{44}^3) + O(K^4)
\end{aligned}
\tag{59}
$$

then from (57) and (59) we have

$$
\begin{aligned}
E &= L^{-1}(-ig_{\mu 4}\dot{x}_\mu + g_{44}) - 1 \\
&= [\{[1 + \tfrac{1}{2}(u^2 + \underset{3}{\gamma}_{\mu\nu} u_\mu u_\nu + 2i\underset{3}{\gamma}_{\mu 4} u_\mu - \underset{3}{\gamma}_{44}) \\
&\quad + \tfrac{3}{8}(u^4 + \underset{3}{\gamma}_{44}^2 + 2u^2 \underset{3}{\gamma}_{44}^2 + 2u^2 \underset{3}{\gamma}_{\mu\nu} u_\nu + 4iu^2 \underset{3}{\gamma}_{\mu 4} u_\mu - 2u^2 \underset{3}{\gamma}_{44} \\
&\quad - 2\underset{3}{\gamma}_{44} \underset{3}{\gamma}_{\mu\nu} u_\mu u_\nu - 4i\underset{3}{\gamma}_{\mu 4} \underset{3}{\gamma}_{44} u_\mu) \\
&\quad + \tfrac{5}{16}(u^6 - 3u^4 \underset{3}{\gamma}_{44} + 3u^2 \underset{3}{\gamma}_{44}^2 - \underset{3}{\gamma}_{44}^3)] \\
&\quad \times [1 - \underset{3}{\gamma}_{44} - i\underset{3}{\gamma}_{\mu 4} u_\mu]\} - 1 + O(K^4)
\end{aligned}
\tag{60}
$$

so that from (38)-(40) we finally have

$$
\begin{aligned}
E = \tfrac{1}{2}u^2 &- V - K_{\sigma\sigma} + \tfrac{3}{2}E^2 + 5V^2 + 6EV + 2EK_{\sigma\sigma} + 6VK_{\sigma\sigma} \\
&- 9VE^2 - 18EV^2 - 2E^3 - 14V^3 - 4W_\sigma^2 - 8W_\mu u_\mu(E + V) \\
&+ \tfrac{1}{2}(4K_{\mu\nu} + E_{\mu\nu})u_\mu u_\nu + \tilde{\mathscr{A}} + O(K^4)
\end{aligned}
\tag{61}
$$

where

$$
\begin{aligned}
\tilde{\mathscr{A}} = -2\pi^{-1}&R(W_{\sigma,\nu}W_{\nu,\sigma}) + 4R(\rho K_{\sigma\sigma}) - 4R(VT^{\sigma\sigma}) \\
&+ \pi^{-1}R(V_{,\sigma\nu}K_{\sigma\nu}) + (4\pi)^{-1}R(V_{,\sigma\nu}E_{\sigma\nu}) \\
&- \pi^{-1}R(VK_{\sigma\nu,\sigma\nu}) - (4\pi)^{-1}R(VE_{\sigma\nu,\sigma\nu}) \\
&+ (2\pi)^{-1}R(VV_{,\sigma}^2) - 4R(\rho V^2)
\end{aligned}
\tag{62}
$$

Equation (61) is the integral of energy of equations (34) when the field (43) is stationary. It has been obtained with an error $O(K^4)$ and contains the integral of energy both in the lower approximations and in the static case (Gambi, 1985; Gambi *et al.*, 1987).

To obtain the integral of angular momentum, we have to add the hypothesis (44). If $0x_3$ is the axis of symmetry, we have

$$x_1 \frac{\partial L}{\partial \dot{x}_2} - x_2 \frac{\partial L}{\partial \dot{x}_1} = -A \tag{63}$$

or, which is the same,

$$L^{-1}[x_1 g_{2\mu} \dot{x}_\mu - x_2 g_{1\mu} \dot{x}_\mu + i(x_1 g_{24} - x_2 g_{14})] = A \tag{64}$$

Now, taking into account (51), we have

$$\begin{aligned} L^{-1} = 1 + \tfrac{1}{2}(u^2 + 2u^2 V - 8 W_\mu u_\mu - 2V + 2K_{\sigma\sigma} - 2V^2) \\ + \tfrac{3}{8}(u^4 + 4V^2 + 4u^2 V) + O(K^3) \end{aligned} \tag{65}$$

and

$$\begin{aligned} &x_1 g_{2\mu} \dot{x}_\mu - x_2 g_{1\mu} \dot{x}_\mu + i(x_1 g_{24} - x_2 g_{14}) \\ &\quad = (1 + 2V - 2K_{\sigma\sigma} + 2V^2)(x_1 u_2 - x_2 u_1) \\ &\quad\quad + 4(x_1 K_{2\mu} u_\mu - x_2 K_{1\mu} u_\mu) + (x_1 E_{2\mu} u_\mu - x_2 E_{1\mu} u_\mu) \\ &\quad\quad + 4(x_2 W_1 - x_1 W_2) + 4(x_2 F_1 - x_1 F_2) + O(K^{7/2}) \end{aligned} \tag{66}$$

so that, from (64)-(66), we have

$$\begin{aligned} A = {} & (1 + 3V + \tfrac{9}{2}V^2 - K_{\sigma\sigma} + \tfrac{1}{2}u^2 + \tfrac{7}{2}u^2 V - 4 W_\mu u_\mu + \tfrac{3}{8}u^4) \times (x_1 u_2 - x_2 u_1) \\ & + (4 + 4V + 2u^2)(x_2 W_1 - x_1 W_2) \\ & + 4(x_1 K_{2\mu} u_\mu - x_2 K_{1\mu} u_\mu) + (x_1 E_{2\mu} u_\mu - x_2 E_{1\mu} u_\mu) \\ & + 4(x_2 F_1 - x_1 F_2) + O(K^{7/2}) \end{aligned} \tag{67}$$

If the field is stationary, then, using (38)-(44), we have from (67)

$$\begin{aligned} A = {} & (1 + 3V + \tfrac{9}{2}V^2 - K_{\sigma\sigma} + \tfrac{1}{2}u^2 + \tfrac{7}{2}u^2 V \\ & - 4 W_\mu u_\mu + \tfrac{3}{8}u^4)(x_1 u_2 - x_2 u_1) \\ & + (4 + 4V + 2u^2)(x_2 u_1 - x_1 u_2) + R\Psi_1 u \\ & + 4(x_2 F_1 - x_2 F_1) + O(K^{7/2}) \end{aligned} \tag{68}$$

where $\Psi_1 = \Psi_1(r^2, x_3)$ is the antisymmetric function of the general descomposition

$$\Sigma_{ij} = \Psi_1 \delta_{ij} + \Psi_{ij}(r^2, x_3) x_i x_j, \qquad (i, j = 1, 2)$$

for the tensor $\Sigma_{ij} = 4K_{ij} + E_{ij}$. But, since from (61) we have

$$u^2 = 2(E+V) + 2K_{\sigma\sigma} - 10V^2 - 3E^2 - 12VE + O(K^3) \tag{69}$$

then from (68) we have, using cylindrical coordinates,

$$\begin{aligned} A = {} & (1 + 4V + E + 8V^2 + 4EV - 4W_\mu u_\mu + \Psi_1)R^2\dot{\Phi} \\ & + 4(1 + E + 2V)(x_2 W_1 - x_1 W_2) \\ & + 4(x_2 F_1 - x_1 F_2) + O(K^{7/2}) \end{aligned} \tag{70}$$

so that, setting $h = A - AE$, we finally have

$$\begin{aligned} h = {} & (1 + 4V + 8V^2 - E^2 - 4W_1 u_1 - 4W_2 u_2 + \Psi_1)R^2\dot{\Phi} \\ & + 4(1 + 2V)(x_2 W_1 - x_1 W_2) + 4(x_2 F_1 - x_1 F_2) + O(K^{7/2}) \end{aligned} \tag{71}$$

which is the desired integral of angular momentum. As the integral of energy, it corresponds to the fourth approximation, that is, it has been obtained with an error $O(K^{7/2})$ and also contains both those for the lower approximations and for the static case (Gambi, 1985; Gambi *et al.*, 1987).

Now, if the massive body has an equatorial plane of symmetry, then, putting $\xi = 1/r$ and using polar coordinates in this plane, from (61) and (71) we have

$$\begin{aligned} & -\left(\frac{d\xi}{\xi^2 dt}\right)^2 + \frac{1}{\xi^2}\left(\frac{d\Phi}{dt}\right)^2 \\ & = 2(E+V) - 10V^2 + 2K_{\sigma\sigma} - 12EV + 28V^3 + 18E^2V \\ & \quad - 12VK_{\sigma\sigma} + 36EV^2 - 4EK_{\sigma\sigma} - 3E^2 + 4E^3 - 2(E+V)\Psi_1 \\ & \quad + 8W_\sigma^2 + 16(E+V)(W_1 u_1 + W_2 u_2) - \tilde{\mathscr{A}} + O(K^4) \end{aligned} \tag{72}$$

$$\begin{aligned} \frac{d\Phi}{dt} = {} & \xi^2[\{h[1 - 4V + 8V^2 + E^2 + 4(W_1 u_1 + W_2 u_2) - \Psi_1] \\ & + 4(x_1 W_2 - x_2 W_1)(1 - 2V) + 4(x_1 F_2 - x_2 F_1) + O(K^{7/2})\}] \end{aligned} \tag{73}$$

and from (72) and (73) we have

$$\left(\frac{d\xi}{d\Phi}\right)^2 = -F(\xi) \tag{74}$$

where

$$F(\xi) = \xi^2 - h^{-2}[2(E+V) + 6V^2 + 2K_{\sigma\sigma} + 4EV - 3E^2 - 12V^3 + 12VK_{\sigma\sigma}$$
$$+ 100EV^2 + 2(E+V)\Psi_1 - 6E^2V + 4E^3$$
$$- 16h^{-1}(E+V)(x_1W_2 - x_2W_1) + 8W_\sigma^2$$
$$+ 24h^{-1}E^2(x_1W_2 - x_2W_1) - 24h^{-1}K_{\sigma\sigma}(x_1W_2 - x_2W_1)$$
$$- 208h^{-1}V(E+V)(x_1W_2 - x_2W_1)$$
$$+ 96h^{-2}(E+V)(x_1W_2 - x_2W_1)^2$$
$$- 16h^{-1}(E+V)(x_1F_2 - x_2F_1) - \tilde{\mathscr{A}}] + O(K^3) \tag{75}$$

so that (74) is the equation for the trajectories on the equatorial plane for the stationary case. It contains those corresponding to the static case because the new potentials W_1, W_2, F_1, and F_2, which are associated with the rotation of the massive body, appear in this case. The contributions of the other potentials were already derived (Gambi *et al.*, 1987) and now are going to derive the corresponding potentials associated with the rotation. First, let us see the way these potentials work in the former approximation. Then, dropping from (74) all the terms that are $O(K^2)$, we have

$$\left(\frac{d\xi}{d\Phi}\right)^2 + \xi^2 = h^{-2}[2(E+V) + 2P - 2Q + 6V^2 + 4EV - 3E^2]$$
$$-h^{-3}[16(E+V)(x_1W_2 - x_2W_1)]O(K^2) \tag{76}$$

where

$$P = \tfrac{1}{2}\kappa JS_{\sigma\sigma} = -\int S_{\sigma\sigma}(\bar{x}')|\bar{x} - \bar{x}'|^{-1}\, d_4\bar{x} \tag{77}$$

$$Q = \tfrac{1}{2}\kappa J(\rho u_\sigma u_\sigma) = -\int \rho u_\sigma u_\sigma |\bar{x} - \bar{x}'|^{-1}\, d_3\bar{x}' \tag{78}$$

Since far from the massive body we have

$$V = m\xi, \qquad P = p\xi, \qquad Q = q\xi, \qquad (x_1W_2 - x_2W_1) = \tfrac{1}{2}J_3\xi \tag{79}$$

where

$$m = O(K), \qquad p = O(K^2), \qquad q = O(K^2), \qquad J_3 = O(K^{3/2}) \tag{80}$$

with J_3 the x_3 component of the angular momentum of the body, then from

(76) we have

$$\left(\frac{d\xi}{d\Phi}\right)^2+\xi^2=h^{-2}(2m\xi+2p\xi+6m^2\xi^2+4mE\xi+2E-3E^2)$$
$$-h^{-3}(8EJ_3\xi+8mJ_3\xi^2)+O(K^2) \tag{81}$$

that is to say, after derivation,

$$\frac{d^2\xi}{d\Phi^2}+\xi(1-6m^2h^{-2}+8mJ_3h^{-3}$$
$$=h^{-2}(m+p-q+2mE-4h^{-1}EJ_3)+O(K^2) \tag{82}$$

so that the integral of (82) is

$$\xi=\frac{m+p-q+2mE-4h^{-1}EJ_3}{h^2(1-6m^2h^{-2}+8mJ_3h^{-3})}$$
$$\times\left[1+\left(1+\frac{(2E-3E^2)h^2(1-6m^2h^{-2}+8mJ_3h^{-3})}{(m+p-q+2mE-4h^{-1}EJ_3)^2}\right)^{1/2}\right]$$
$$\times\cos[\Phi(1-6m^2h^{-2}+8mJ_3h^{-3})^{1/2}] \tag{83}$$

As can be seen from this solution, the terms m and h^2 appear supplemented by p, q, $2mE$, $4h^{-1}EJ_3$, and $6m^2$, $8mJ_3h^{-1}$ in the first factor, and the terms $2Eh^2/m^2$ and $\cos\Phi$ are supplemented by

$$\frac{(\cdots-3E^2)h^2(\cdots-6m^2h^2+8mJ_3h^{-3})}{(\cdots+p-q+2mE-4h^{-1}EJ_3)^2}$$

and

$$\cos\Phi(\cdots-6m^2h^{-2}+8mJ_3h^{-3})^{1/2}$$

respectively. Also it may be observed that the relations between the relativistic terms are maintained as in the classical equations. In fact, equations (83) correspond to a quasiconic whose semi-latus rectum l is given by

$$l=\frac{h^2(1-6m^2h^{-2}+8mJ_3h^{-3})}{m+p-q+2mE-4h^{-1}EJ_3} \tag{84}$$

and whose eccentricity is

$$e=1+\left(\frac{2E-3E^2)h^2(1-6m^2h^2+8mJ_3h^{-3}}{(m+p-q+2mE-4h^{-1}EJ_3)^2}\right)^{1/2} \tag{85}$$

Since this eccentricity is greater, equal to, or less than one if the terms between parentheses in (85) are greater, equal to, or less than zero, respectively, it is clear that this fact only depends on the value taken by the factor $E-3/2E^2$, which is the total energy of the particle, because if we taken into account that

$$E=(\tfrac{1}{2}u^2-V)-K_{\sigma\sigma}+\tfrac{3}{8}u^4+\tfrac{3}{2}u^4+\tfrac{3}{2}u^2V+\tfrac{1}{2}V^2+O(K^3) \tag{86}$$

then, from (61) we have

$$E-\tfrac{3}{2}E^2=\tfrac{1}{2}u^2(1+6V)-V-V^2-K_{\sigma\sigma} \tag{87}$$

for this approximation. Therefore, the trajectories are of elliptic, parabolic, or hyperbolic type if $E-\frac{3}{2}E^2$ is less than, equal to, or greater than zero, respectively, and so the higher order corrections do not modify essentially this classification.

4. CONTRIBUTIONS DUE TO ROTATION

Since, in accord with (58), in the stationary gravitational models we are considering the Lagrangian is given by

$$\begin{aligned}L=-[1&-(2V+u^2)+(2V^2-2Vu^2+8W_\mu u_\mu-2K_{\sigma\sigma})\\&+(-4V^3-2V^2u^2+4VK_{\sigma\sigma}+2K_{\sigma\sigma}u^2-8W_\sigma^2\\&-\Sigma_{\mu\nu}u_\mu u_\nu+8F_\mu u_\mu+2\tilde{\mathscr{A}})+O(K^4)]^{1/2}\end{aligned} \tag{88}$$

then for this approximation we have

$$\begin{aligned}L=-1+V&+\frac{u^2}{2}-\frac{V^2}{2}+\tfrac{3}{2}Vu^2-4W_\mu u_\mu+K_{\sigma\sigma}+\frac{u^4}{8}\\&+\tfrac{3}{2}V^3+\tfrac{9}{4}V^2u^2-K_{\sigma\sigma}V-\tfrac{1}{2}K_{\sigma\sigma}u^2-5VW_\mu u_\mu+4W_\sigma^2\\&+\tfrac{1}{2}\Sigma_{\mu\nu}u_\mu u_\nu+\tfrac{7}{8}Vu^4-2u^2W_\mu u_\mu-4F_\mu u_\mu-\tilde{\mathscr{A}}+\frac{u^6}{16}\end{aligned} \tag{89}$$

so that for the generalized momentum P_α we have

$$\begin{aligned}p_\alpha=u_\alpha&+3Vu_\alpha-4W_\alpha+\tfrac{1}{2}u^2u_\alpha+\tfrac{9}{2}V^2u_\alpha-K_{\sigma\sigma}u_\alpha\\&-4VW_\alpha+\Sigma_{\mu\alpha}u_\mu+\tfrac{7}{2}Vu^2u_\alpha-4W_\mu u_\mu u_\alpha-2u^2W_\alpha\\&-4F_\alpha+\tfrac{3}{8}u^4u_\alpha\end{aligned} \tag{90}$$

and for the Hamiltonian H we have

$$\begin{aligned}H=1&+\frac{u^2}{2}-V+\tfrac{1}{2}V^2-K_{\sigma\sigma}+\tfrac{3}{2}Vu^2-\tfrac{1}{2}K_{\sigma\sigma}u^2+\tfrac{9}{4}V^2u^2\\&+\tfrac{1}{2}\Sigma_{\mu\alpha}u_\mu u_\alpha-4W_\sigma^2+\tfrac{3}{8}u^4+VK_{\sigma\sigma}+\tfrac{21}{8}Vu^4-\tfrac{3}{2}V^3\\&+\tilde{\mathscr{A}}-4u^2W_\mu u_\mu+\frac{5}{16}u^6\end{aligned} \tag{91}$$

But, taking into account that

$$u_\alpha = p_\alpha - 3Vp_\alpha + 4W_\alpha - \tfrac{1}{2}p^2 p_\alpha + \tfrac{9}{2}V^2 p_\alpha + \tfrac{5}{2}Vp^2 p_\alpha - 8VW_\alpha + K_{\sigma\sigma}p_\alpha + \tfrac{3}{8}p^4 p_\alpha - 4p^2 W_\alpha + 4W_\mu p_\mu p_\alpha - \Sigma_{\mu\alpha}p_\mu + 4F_\sigma \qquad (92)$$

then, from (91) and (92), we finally have

$$H = 1 + \frac{p^2}{2} - V - \tfrac{3}{2}Vp^2 + 4W_\alpha p_\alpha - \frac{p^4}{8} + \tfrac{1}{2}V^2 - K_{\sigma\sigma} + \tfrac{9}{4}V^2p^2 + \tfrac{5}{8}Vp^4 - 8VW_\alpha p_\alpha + \tfrac{1}{2}K_{\sigma\sigma}p^2 - \tfrac{1}{2}\Sigma_{\mu\alpha}p_\mu p_\alpha + 4F_\alpha p_\alpha + 4W_\alpha^2 + \frac{1}{16}p^6 + VK_{\sigma\sigma} - \tfrac{3}{2}V^3 + \tilde{\mathscr{A}} + O(K^4) \qquad (93)$$

so that the Hamilton–Jacobi equation, which in this case is

$$H(x_\mu, \partial S/\partial x_\mu) = E \qquad (94)$$

has the following form:

$$E = \tfrac{1}{2}p^2 - V(1 + 3V + 6V^2 - 2K_{\sigma\sigma} - 16EV - 2E^2) - K_{\sigma\sigma} + 4W_\alpha^2 + \tilde{\mathscr{A}} - \tfrac{1}{2}\Sigma_{\mu\nu}p_\mu p_\nu + 4[(1 - \tfrac{11}{2})W_\alpha + F_\alpha]p_\alpha + O(K^4) \qquad (95)$$

as can be seen if (93) and the fact that

$$\tfrac{1}{2}p^2 = V + 3V^2 + 4EV + \tfrac{1}{2}E + K_{\sigma\sigma} + 4W_\alpha p_\alpha + O(K^3) \qquad (96)$$

are taken into account.

Now, let us suppose that the field is generated by a massive body in steady rotation as described before and that the particle is in the plane of its equator. Since in spherical coordinates

$$p_r = \frac{\partial S}{\partial r}, \qquad P_\Phi = \frac{1}{r}\frac{\partial S}{\partial \Phi}, \qquad p_\theta = 0 \qquad (97)$$

and since the only nonnull components of $\Sigma_{\mu\nu}p_\mu p_\nu$ in (95) are

$$\Sigma_{11}p_1p_1 + 2\Sigma_{12}p_1p_2 + \Sigma_{22}p_2p_2 \qquad (98)$$

with $p_1 = \cos\Phi\, p_r - \sin\Phi\, p_\Phi$ and $p_2 = \sin\Phi\, p_4 + \cos\Phi\, p_\Phi$, then if we take into account that

$$\Sigma_{11} = \Psi_1 + \Psi_2 r^2\cos^2\Phi, \qquad \Sigma_{22} = \Psi_1 + \Psi_2 r^2 \sin^2\Phi$$
$$\Sigma_{12} = \Sigma_{21} = \tfrac{1}{2}\Psi_2 r^2 \sin 2\Phi \qquad (99)$$

we have

$$\Sigma_{\mu\nu}p_\mu p_\nu = (\Psi_1 + \Psi_2 r^2)p_r^2 + \Psi_1 p_\Phi^2 \tag{100}$$

On the other hand, since far from the massive body we have (79), then

$$W_r = W_1 \cos\Phi \sin\theta + W_2 \sin\Phi \sin\theta + W_3 \cos\theta = 0$$

$$W_\Phi = -W_1 \sin\Phi + W_2 \cos\Phi = \frac{1}{2r^3} J_3$$

$$W_\theta = -W_1 \cos\Phi \cos\theta - W_2 \sin\Phi \cos\theta + W_3 \sin\theta = 0 \tag{101}$$

and

$$F_\mu = J(f_\mu) = -\frac{1}{4\pi r}\int_\infty f_\mu(\bar{x}')\, d_3\bar{x}' + O(r^{-2}\log r) \tag{102}$$

$$\equiv -\frac{\mathscr{F}_\mu}{4\pi r} + O(r^{-2}\log r) \tag{102'}$$

where

$$f_\mu = (2\pi)^{-1}[(V_{,\mu} W_{\sigma,\mu} - V_{,\mu\sigma} W_\sigma) + (W_\mu \Delta V - V\Delta W_\mu)]$$

and $F_r = F_r$, $F_\Phi = 0$, and $F_\theta = 0$, so that, taking into account (97), (100), and (101), from (95) we finally have

$$\begin{aligned} E = {} & \frac{1}{2}\left[\left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_0}{\partial \Phi}\right)^2\right] - V(1 + 3V + 6V^2 - 2K_{\sigma\sigma} - 16EV - 2E^2) \\ & - K_{\sigma\sigma} + 4W_\Phi^2 + 2\tilde{\mathscr{A}} + 4W_\Phi\left(1 - \frac{11}{2}V\right)\frac{1}{r}\left(\frac{\partial S_0}{\partial \Phi}\right) \\ & + 4F_r\left(\frac{\partial S_0}{\partial r}\right) + (\Psi_1 + r^2\Psi_2)\left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{r^2}\Psi_1\left(\frac{\partial S_0}{\nabla \Phi}\right)^2 + O(K^4) \end{aligned} \tag{103}$$

where $S_0 = S + Et$.

Now, writing S_0 in the form

$$S_0(r, \Phi) = S_r(r) + S_\Phi(\Phi) \tag{104}$$

and taking into account that Φ is a cyclic variable, we have

$$\partial S_0/\partial\Phi \equiv p_\Phi \equiv h \tag{105}$$

so that (103) is equivalent to

$$\begin{aligned} E = {} & \tfrac{1}{2}\left(p_r^2 + \frac{h^2}{r^2}\right) - V(1 + 3V + 6V^2 - 2K_{\sigma\sigma} - 16EV - 2E^2) \\ & - K_{\sigma\sigma} + 4W_\Phi^2 + 2\tilde{\mathscr{A}} + 4\left[W_\Phi\left(1 - \frac{11}{2}V\right)\right]\frac{h}{r} + 4F_r p_r \\ & + (\Psi_1 + r^2\Psi_2)p_r^2 + \frac{1}{r^2}h^2\Psi_1 + O(K^4) \end{aligned} \tag{106}$$

But, since in the lowest approximation we have $p_r = [2(E+V) - h^2/r^2]^{1/2}$, then from (106) we have

$$p_r = \Big[2E(1-\Psi_1 - r^2\Psi_2) + 2V(1+3V+6V^2 - 2K_{\sigma\sigma} - 16EV - 2E^2 - \Psi_1 - r^2\Psi_2) + 2K_{\sigma\sigma} - 8W_\Phi^2 - 4\tilde{\mathscr{A}} - 8W_\Phi\left(1-\frac{11}{2}V\right)\frac{h^2}{r^2} - 8F_r\left[2(E+V)-\frac{h^2}{r^2}\right]^{1/2} - (1-2r^2\Psi_2)\frac{h^2}{r^2}\Big] \tag{107}$$

so that, finally,

$$S_r = \int^r \Big[2E(1-\Psi_1 - r^2\Psi_2) - (1-2r^2\Psi_2)\frac{h^2}{r^2} - 8W_\Phi\left(1-\frac{11}{2}V\right)\frac{h^2}{r^2} + 2V(1+3V+6V^2-2K_{\sigma\sigma} - 16EV - 2E^2 - \Psi_1 - r^2\Psi_2) + 2K_{\sigma\sigma} - 8W_\Phi^2 - 4\tilde{\mathscr{A}} - 8F_r\left\{\left[2(E+V)-\frac{h^2}{r^2}\right]^{1/2}\right\}^{1/2} dr \tag{108}$$

and

$$t+\tau = \frac{\partial S}{\partial E} = \int \frac{1}{p_r} A2(1-\Psi_1 - r^2\Psi_2) - 16V^2 - 4EV - 4F_r[2(E+V) - \frac{h^2}{r^2}\Big]^{1/2}\Big\}\, dr \tag{109}$$

$$\Phi + \alpha = -\frac{\partial S}{\partial h} = \int \frac{1}{p_r}\left\{(1-2r^2\Psi_2)\frac{h}{r^2} + 8W_\Phi\left(1-\frac{11}{2}V\right)\frac{1}{r^2} - 4F_r\frac{h}{r^2}\left[2(E+V)-\frac{h^2}{r^2}\right]^{1/2}\right\} dr \tag{110}$$

As can be seen in (108), the rotation manifests itself through the terms $-8W_\Phi(1-\frac{11}{2}V)h/r$, $8W_\Phi^2$, $-8F_r[2(E+V)-h^2/r^2]^{1/2}$, and $8\pi^{-1}R(W_{\sigma,\mu})^2$, which is in $\tilde{\mathscr{A}}$. In order to obtain their contributions, we write (110) in the form

$$\Phi + \alpha = -\frac{\partial}{\partial h}\int\left[2(E+V+\delta V) - \frac{h^2}{r^2}\right]^{1/2} dr \tag{111}$$

where

$$\delta V = \delta \underset{3}{V}^{(\text{stat})} + \delta \underset{2}{V}^{(\text{rot})} + \delta \underset{3}{V}^{(\text{rot})} \tag{112}$$

so that $\underset{3}{\delta V}^{(\text{stat})}$ contains all the terms in (107) already considered in the static case and $\underset{2}{\delta V}^{(\text{rot})}$ and $\underset{3}{\delta V}^{(\text{rot})}$ contain the terms associated with the rotation of second and third order, respectively. Then, developing (112), we have

$$\left[2(E+V+\delta V)-\frac{h^2}{r^2}\right]^{1/2}=\left[2(E+V)-\frac{h^2}{r^2}\right]^{1/2}+\frac{1}{2}\,\delta V\left[2(E+V)-\frac{h^2}{r^2}\right]^{-1/2}$$
$$-\frac{1}{8}(\delta V)^2\left[2(E+V)-\frac{h^2}{r^2}\right]^{-3/2}+O[(\delta V)^3] \quad (113)$$

Now, since for an orbit of elliptic type with apsidal distances at r_1 and r_2 we have the advance

$$\Delta\Phi=-2\frac{\partial}{\partial h}\int_{r_1}^{r_2}\left[2(E+V+\delta V)-\frac{h^2}{r^2}\right]^{1/2}dr \quad (114)$$

then, writing $\Delta\Phi$ in the form $\Delta\Phi^{(\text{stat})}+\Delta\Phi^{(\text{rot})}$ and taking into account (112) and (113), from (114) we have

$$\Delta\Phi^{(\text{rot})}=-\frac{\partial}{\partial h}\left\{\frac{1}{h}\int_0^{\pi} r^2\underset{2}{\delta V}^{(\text{rot})}d\Phi+\frac{1}{h}\int_0^{\pi} r^2\underset{3}{\delta V}^{(\text{rot})}\,d\Phi\right.$$
$$\left.+\frac{1}{4}\int_0^{\pi} r^4(\underset{2}{\delta V}^{(\text{rot})})^2[2(E+V)r^2-h^2]^{-1}\,d\Phi\right\}+O[(\delta V)^3] \quad (115)$$

The advance $\underset{2}{\Delta\Phi}^{(\text{rot})}$ is known. In fact, taking into account (101), we have

$$\underset{2}{\delta V}^{(\text{rot})}=4hJ_3/r^3 \quad (116)$$

so that

$$\underset{2}{\Delta\Phi}^{(\text{rot})}=-\frac{\partial}{\partial h}\left(\frac{1}{h}\int_0^{\pi} r^2 4\frac{hJ_3}{r^3}\,d\Phi\right) \quad (117)$$

But, since in first approximation $l=a(1-e^2)$, then we have

$$\underset{2}{\Delta\Phi}^{(\text{rot})}=\frac{8\pi J_3 h}{a^2(1-e^2)^2}=\frac{8\pi J_3 m}{a^{3/2}(1-e^2)^{3/2}} \quad (118)$$

which corresponds to an an angular velocity of precession of the orbit given by

$$\Omega=\frac{4J_3 m^{3/2}}{a^{3/2}}(1-e^2)^{3/2} \quad (119)$$

that is, the corresponding one to the Lense-Thirring effect of rotation.

The perturbation $\delta \underset{3}{V}^{(\text{rot})}$ is given by

$$\delta \underset{3}{V}^{(\text{rot})} = 44 V W_\Phi \frac{h}{r} - 8 W_\Phi^2 + \frac{2}{\pi} \int_\infty \frac{W_{\nu,\sigma} W_{\nu,\sigma}}{|\bar{x} - \bar{x}'|} d_3 \bar{x}$$
$$- 8 F_r \left[2(E + V) - \frac{h^2}{r^2} \right]^{1/2} \tag{120}$$

but because when $r \to \infty$ the integrand in (119) falls like □, then we have

$$\frac{2}{\pi} \int_\infty \frac{W_{\nu,\sigma} W_{\nu,\sigma}}{|\bar{x} - \bar{x}'|} d_3 \bar{x}' = \frac{2}{\pi r} \int_\infty W_{\nu,\sigma} W_{\nu,\sigma} \, d\bar{x}' + O(r^{-2} \log r)$$
$$\equiv \frac{2 \mathcal{W}}{\pi r} + O(r^{-2} \log r) \tag{121}$$

so that, taking into account (101) and (102), we have

$$\delta \underset{3}{V}^{(\text{rot})} = 22 \frac{m J_3 h}{r^4} - 2 \frac{J_3^2}{r^4} + \frac{4 \mathcal{W}}{2r} - 8 \frac{\mathcal{F}}{r^2} (2Er^2 + 2mr - h^2)^{1/2} \tag{122}$$

and, therefore, carrying (121) to (115), we have

$$\Delta \underset{3}{\Phi}^{(\text{rot})} = -\frac{\partial}{\partial h} \left\{ \frac{22 m^3 J_3 h - 2 J_3^2 m^2}{h^5} \int_0^\pi (1 + e \cos \Phi)^2 \, d\Phi \right.$$
$$+ \frac{4}{\pi} \frac{\mathcal{W} h}{m} \int_0^\pi (1 + e \cos \Phi)^{-1} \, d\Phi - 8 \frac{F}{h} \int_0^\pi \left[2E \frac{h^4}{m^2} (1 + e \cos \Phi)^{-2} \right.$$
$$\left. \left. + 2h^2 (1 + e \cos \Phi)^{-1/2} h^2 \right]^{1/2} d\Phi \right\} \tag{123}$$

from which, taking into account (118), we finally have

$$\Delta \Phi^{(\text{rot})} = \frac{8 J_3 m}{a^{3/2} (1 - e^2)^{3/2}} - 4 \frac{\mathcal{W}}{m} (1 - e^2)^{-1/2} - 16 \frac{\mathcal{F} e}{(1 + e^2)^2}$$
$$+ \pi \left(1 + \frac{e^2}{2} \right) \left[\frac{88 m^{1/2} J_3}{[a(1 - e^2)]^{5/2}} - \frac{10 J_3^2}{m [a(1 - e^2)]^3} \right] \tag{124}$$

which is the value for the total advance.

REFERENCES

Gambi, J. M. (1983). *General Relativity and Gravitations*, **13**, 397.

Gambi, J. M. (1985). *International Journal of Theoretical Physics*, **24**, 467.

Gambi, J. M., and San Miguel, A. (1986). *International Journal of Theoretical Physics*, **25**, 183.

Gambi, J. M., San Miguel, A., and Vicente, F. (1987). *International Journal of Theoretical Physics*, **26**,649.

Hogan, P. A., and McCrea, J. D. (1974). *General Relativity and Gravitation*, **5**, 79.

McCrea, J. D. (1981). *General Relativity and Gravitation*, **13**, 397.

McCrea, J. D., and O'Brien, G. M. (1978). *General Relativity and Gravitation*, **9**, 1101.

O'Brien, G. M. (1979). *General Relativity and Gravitation*, **10**, 129.

Synge, J. L. (1970). *Proceedings of the Royal Irish Academy A* **69**, 11.